Exploiting loops in the graph of trifocal tensors for calibrating a network of cameras
Supplemental material

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Abstract. Here we present rigorous proofs of some results stated in the paper.

1 Proof of Proposition 1

We begin by considering the case where all the 3 rows of the fundamental matrices $\mathtt{F}^{ij}$ and $\mathtt{F}^{ik}$ are different from the zero vector of $\mathbb{R}^3$. This implies that the columns of the matrices $\mathtt{A}_t^{ij}$ and $\mathtt{A}_t^{ij}$ form two orthogonal bases of $\mathbb{R}^3$. (Indeed, it is well-known that the epipole $\mathbf{e}^{ij}$ is orthogonal to the rows of $\mathtt{F}^{ij}$, while $(\mathtt{F}^{ij}_{t,1:3})^\top \times \mathbf{e}^{ij}$ is orthogonal to $\mathtt{F}^{ij}_{t,1:3}$ and to $\mathbf{e}^{ij}$ by virtue of the definition of the vector product.) Therefore, $\mathtt{A}_t^{ik}$ and $\mathtt{A}_t^{ij}$ are invertible. Let us define

$$\begin{bmatrix} a_t & b_t & c_t \\ d_t & e_t & f_t \\ g_t & h_t & i_t \end{bmatrix} = (\mathtt{A}_t^{ij})^{-1}\mathtt{T}_t^{ijk}(\mathtt{A}_t^{ik})^{-\top}. \tag{1}$$

Let us show that $a_t = b_t = c_t = 0$. Recall that the matrix $\mathtt{T}_t^{ijk}$ relates a point $\mathbf{p} = (p_1, p_2, p_3) \in \mathbb{P}^2$ to its epipolar line $\mathbf{l} = \mathtt{F}^{ij^\top}\mathbf{p}$ through the equation $\mathbf{l}^\top \sum_{s=1}^3 p_s \mathtt{T}_s^{ijk} = \mathbf{0}^\top$ [1, p. 373]. Choosing as $\mathbf{p}$ the vector $(\delta_{t1}, \delta_{t2}, \delta_{t3})^\top$, where $\delta_{t\ell}$ stands for the Kronecker symbol that equals one if $t = \ell$ and zero otherwise, we get $\mathtt{F}^{ij}_{t,1:3}\mathtt{T}_t^{ijk} = \mathbf{0}^\top$. This equation, in conjunction with (1), the definition of $\mathtt{A}_t^{ij}$ and the invertibility of $\mathtt{A}_t^{ik}$ entails that

$$\begin{bmatrix} a_t & d_t & g_t \\ b_t & e_t & h_t \\ c_t & f_t & i_t \end{bmatrix} \begin{bmatrix} \|(\mathtt{F}^{ij}_{t,1:3})^\top\|^2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This yields $a_t = b_t = c_t = 0$. By a symmetric argument, we also check that $d_t = g_t = 0$. Thus, the Trifocal Tensor necessarily reduces to the form:

$$\mathtt{T}_t^{ijk} = \mathtt{A}_t^{ij} \begin{bmatrix} 0 & 0 & 0 \\ 0 & e_t & f_t \\ 0 & h_t & i_t \end{bmatrix} (\mathtt{A}_t^{ik})^\top \tag{2}$$

Since the rank of any fundamental matrix is equal to two, there exists an index $t' \in \{1, 2, 3\}$ such that $\mathtt{F}^{ij}_{t',1:3}$ and $\mathtt{F}^{ij}_{t,1:3}$ are not collinear. We have already checked that

$\mathtt{F}^{ij}_{t,1:3}\mathtt{T}^{ijk}_t = \mathbf{0}^\mathsf{T}$ and, similarly, $\mathtt{F}^{ij}_{t',1:3}\mathtt{T}^{ijk}_{t'} = \mathbf{0}^\mathsf{T}$. Therefore, substituting $\mathbf{p} = (\delta_{t1}, \delta_{t2}, \delta_{t3})^\mathsf{T} + (\delta_{t'1}, \delta_{t'2}, \delta_{t'3})^\mathsf{T}$ in the equation $\mathbf{p}^\mathsf{T}\mathtt{F}^{ij}\sum_{s=1}^3 p_s \mathtt{T}^{ijk}_s = 0$, we get

$$\mathtt{F}^{ij}_{t',1:3}\mathtt{T}^{ijk}_t + \mathtt{F}^{ij}_{t,1:3}\mathtt{T}^{ijk}_{t'} = \mathbf{0}^\mathsf{T}. \tag{3}$$

Now, let us observe that $(\mathtt{F}^{ij}_{t',1:3})^\mathsf{T}((\mathtt{F}^{ij}_{t,1:3})^\mathsf{T} \times \mathbf{e}^{ij}) = -(\mathtt{F}^{ij}_{t,1:3})^\mathsf{T}((\mathtt{F}^{ij}_{t',1:3})^\mathsf{T} \times \mathbf{e}^{ij}) := \beta_{t,t'}$. Moreover, $\beta_{t,t'} \neq \mathbf{0}$ since the vectors $\mathtt{F}^{ij}_{t,1:3}$ and $\mathtt{F}^{ij}_{t',1:3}$ are linearly independent and orthogonal to $\mathbf{e}^{ij}$. This observation together with (2) and (3) leads to

$$\mathtt{A}^{ik}_t \begin{bmatrix} 0 & 0 & 0 \\ 0 & e_t & h_t \\ 0 & f_t & i_t \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ 0 \end{bmatrix} + \mathtt{A}^{ik}_{t'} \begin{bmatrix} 0 & 0 & 0 \\ 0 & e_{t'} & h_{t'} \\ 0 & f_{t'} & i_{t'} \end{bmatrix} \begin{bmatrix} \alpha \\ -\beta \\ 0 \end{bmatrix} = \mathbf{0},$$

where we have used the shorthands $\alpha = \alpha_{t,t'}$ and $\beta = \beta_{t,t'}$. Matrix multiplication yields

$\beta_{t,t'}\Big(\big[0,\ e_t,\ f_t\big]\,(\mathtt{A}^{ik}_t)^\mathsf{T} - \big[0,\ e_{t'},\ f_{t'}\big]\,(\mathtt{A}^{ik}_{t'})^\mathsf{T}\Big) = \mathbf{0}^\mathsf{T}$.

The last display is equivalent to

$(e_t\mathtt{F}^{ik}_{t,1:3} - e_{t'}\mathtt{F}^{ik}_{t',1:3})^\mathsf{T} \times \mathbf{e}^{ik} + (f_t - f_{t'})\mathbf{e}^{ik} = \mathbf{0}$,

which is possible if and only if $f_t = f_{t'}$ and $e_t\mathtt{F}^{ik}_{t,1:3} - e_{t'}\mathtt{F}^{ik}_{t',1:3} = \mathbf{0}$. Since $\mathtt{F}^{ik}_{t,1:3}$ and $\mathtt{F}^{ik}_{t',1:3}$ are linearly independent, we conclude that $e_t = e_{t'} = 0$. In addition, using a symmetric argument, we get $h_t = h_{t'}$ and thus

$$\mathtt{T}^{ijk}_t = \mathtt{A}^{ij}_t \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & f_t \\ 0 & h_t & i_t \end{bmatrix} (\mathtt{A}^{ik}_t)^\mathsf{T}, \mathtt{T}^{ijk}_{t'} = \mathtt{A}^{ij}_{t'} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & f_t \\ 0 & h_t & i_{t'} \end{bmatrix} (\mathtt{A}^{ik}_{t'})^\mathsf{T}$$

Let now t'' be the element of the index set $\{1,2,3\}$ that is different from t and t'. Repeating the same arguments with t replaced by t'' leads to a formula similar to (4) in which t' is replaced by t'' everywhere. The first claim of the proposition follows by dividing all the entries of $[\mathtt{T}^{ijk}]$ by f_t, which is $\neq 0$ since otherwise the fundamental matrix computed from this trifocal tensor is of rank < 2, which is impossible.

In the case where the matrix $\mathtt{F}^{ij}$ or $\mathtt{F}^{ik}$ contain zero rows, one can merely use the fact that Eq. (3) (see page 4 of the paper) characterize all the triplets of camera matrices (up to a projective homography) which are compatible with the fundamental matrices $\mathtt{F}^{ij}$ and $\mathtt{F}^{ik}$. In view of [1, Eq. 15.1], the trifocal tensor corresponding to these camera matrices coincides with the one defined in the statement of the proposition. This completes the proof.

2 Proof of Equation (5)

Let $\mathtt{P}^i$ and $\mathtt{P}^k$ be the camera matrices computed from the triplet (i,j,k) with parameter $\boldsymbol{\gamma}$ and let $\bar{\mathtt{P}}^i$ and $\bar{\mathtt{P}}^k$ be those computed from the triplet (k,i,ℓ) with parameter $\boldsymbol{\gamma}'$. Thus, we are looking for a homography $\mathtt{H}$ such that

$$[\mathtt{I}_{3\times3}|\mathbf{0}]\mathtt{H} \cong \mathrm{kron}([\boldsymbol{\gamma}'_{1:3}, 1], \mathbf{e}^{ki}) - \big[[\mathbf{e}^{ki}]_\times \mathtt{F}^{ik}|\mathbf{0}\big], \tag{4}$$

$$\big[\gamma_0[\mathbf{e}^{ik}]_\times \mathtt{F}^{ki}|\mathbf{e}^{ik}\big]\mathtt{H} \cong [\mathtt{I}_{3\times3}|\mathbf{0}], \tag{5}$$

where $\cong$ stands for the proportionality. The first equation yields $\mathtt{H}_{1:3,1:4} = \big[\mathrm{kron}(\boldsymbol{\gamma}'_{1:3}, \mathbf{e}^{ki}) - [\mathbf{e}^{ki}]_\times \mathtt{F}^{ik} | \mathbf{e}^{ki}\big]$. Inserting this in (5) and using $\mathtt{F}^{ki}\mathbf{e}^{ki} = \mathbf{0}$, we get

$$-\gamma_0[\mathbf{e}^{ik}]_\times \mathtt{F}^{ki}[\mathbf{e}^{ki}]_\times \mathtt{F}^{ik} + \mathbf{e}^{ik}\mathtt{H}_{4,1:3} = \alpha \mathtt{I}_{3\times 3} \tag{6}$$

and $\mathbf{e}^{ik}\mathtt{H}_{4,4} = 0$. This implies that $\mathtt{H}_{4,4} = 0$. Furthermore, multiplying both sides of (6) by $(\mathbf{e}^{ik})^\mathsf{T}$, we get $\mathtt{H}_{4,1:3} = \alpha(\mathbf{e}^{ik})^\mathsf{T}$. To complete the proof, it remains to determine the value of α. This is done by computing the trace of both sides in (6).

3 Proof of Proposition 2

In view of Eq. (12) of the paper, it holds

$$\mathtt{P}^{2,\boldsymbol{\gamma}_i}\mathtt{H}^{v_i} \cong \mathtt{P}^{i+1,*}, \qquad \mathtt{P}^{3,\boldsymbol{\gamma}_i}\mathtt{H}^{v_i} \cong \mathtt{P}^{i+2,*} \tag{7}$$

$$\mathtt{P}^{1,\boldsymbol{\gamma}i+1}\mathtt{H}^{v_{i+1}} \cong \mathtt{P}^{i+1,*}, \qquad \mathtt{P}^{2,\boldsymbol{\gamma}i+1}\mathtt{H}^{v_{i+1}} \cong \mathtt{P}^{i+2,*}. \tag{8}$$

Therefore,

$$\mathtt{P}^{2,\boldsymbol{\gamma}_i}\mathtt{H}^{v_i} \cong \mathtt{P}^{1,\boldsymbol{\gamma}i+1}\mathtt{H}^{v_{i+1}}, \qquad \mathtt{P}^{3,\boldsymbol{\gamma}_i}\mathtt{H}^{v_i} \cong \mathtt{P}^{2,\boldsymbol{\gamma}i+1}\mathtt{H}^{v_{i+1}}. \tag{9}$$

Furthermore, by virtue of Eq. (11) of the paper,

$$\mathtt{P}^{1,\boldsymbol{\gamma}_{i+1}} \cong \mathtt{P}^{2,\boldsymbol{\gamma}_i}\mathtt{H}^{i,i+1}, \quad \mathtt{P}^{2,\boldsymbol{\gamma}_{i+1}} \cong \mathtt{P}^{3,\boldsymbol{\gamma}_i}\mathtt{H}^{i,i+1}. \tag{10}$$

Substituting (10) in (9), we get

$$\mathtt{P}^{2,\boldsymbol{\gamma}_i}\mathtt{H}^{v_i} \cong \mathtt{P}^{2,\boldsymbol{\gamma}_i}\mathtt{H}^{i,i+1}\mathtt{H}^{v_{i+1}}, \qquad \mathtt{P}^{3,\boldsymbol{\gamma}_i}\mathtt{H}^{v_i} \cong \mathtt{P}^{3,\boldsymbol{\gamma}_i}\mathtt{H}^{i,i+1}\mathtt{H}^{v_{i+1}}. \tag{11}$$

If the centers of $\mathtt{P}^{2,\boldsymbol{\gamma}_i}$ and $\mathtt{P}^{3,\boldsymbol{\gamma}_i}$ differ, which is equivalent to $\mathtt{P}^{i+1,*}, \mathtt{P}^{i+2,*}$ having different centers, then Eq. (11) can be satisfied if and only if $\mathtt{H}^{v_i} \cong \mathtt{H}^{i,i+1}\mathtt{H}^{v_{i+1}}$.

To prove i) and ii), we need to trnasform the relations of proportionality into the relations of equality. Since $\mathtt{H}^{v_i} = \mathtt{H}^{i,i+1}\mathtt{H}^{v_{i+1}}$, there exists a real number $\alpha_i \neq 0$ such that $\mathtt{H}^{v_i} = \alpha_i \mathtt{H}^{i,i+1}\mathtt{H}^{v_{i+1}}$. Thus, we have

$$\mathtt{H}^{v_1} = \alpha_1 \mathtt{H}^{1,2}\mathtt{H}^{v_2}, \tag{12}$$

$$\mathtt{H}^{v_2} = \alpha_2 \mathtt{H}^{2,3}\mathtt{H}^{v_3}, \tag{13}$$

$$\vdots \tag{14}$$

$$\mathtt{H}^{v_n} = \alpha_n \mathtt{H}^{n,1}\mathtt{H}^{v_1}. \tag{15}$$

Let us denote $\tilde{\mathtt{H}}^{v_1} = \mathtt{H}^{v_1}, \tilde{\mathtt{H}}^{v_2} = \alpha_1\mathtt{H}^{v_2}, \tilde{\mathtt{H}}^{v_3} = \alpha_1\alpha_2\mathtt{H}^{v_3}, \ldots, \tilde{\mathtt{H}}^{v_n} = \alpha_1 \times \ldots \times \alpha_{n-1}\mathtt{H}^{v_n}$. Each $\tilde{\mathtt{H}}^{v_i}$ being proportional to $\mathtt{H}^{v_i}$, satisfies Eq. (12) of the paper. This leads to the assertion i) of Proposition 2, since $\tilde{\mathtt{H}}^{v_i} = \mathtt{H}^{i,i+1}\tilde{\mathtt{H}}^{v_{i+1}}$ for $i = 1, \ldots, n-1$. Moreover, we have

$$\tilde{\mathtt{H}}^{v_n} = \Big(\prod_{i=1}^{n} \alpha_i\Big)\mathtt{H}^{n,1}\tilde{\mathtt{H}}^{v_1} \stackrel{\text{notation}}{=} \alpha^{-1}\mathtt{H}^{n,1}\tilde{\mathtt{H}}^{v_1}.$$

This implies that

$$\prod_{i=1}^{n} \tilde{\mathtt{H}}^{v_i} = \Big(\prod_{i=1}^{n-1} \mathtt{H}^{i,i+1}\tilde{\mathtt{H}}^{v_{i+1}}\Big)\Big(\prod_{i=1}^{n}\alpha_i\Big)\mathtt{H}^{n,1}\tilde{\mathtt{H}}^{v_1}$$
$$= \Big(\prod_{i=1}^{n} \mathtt{H}^{i,i+1}\Big)\Big(\prod_{i=1}^{n}\alpha_i\Big)\prod_{i=1}^{n}\tilde{\mathtt{H}}^{v_i},$$

which is equivalent to

$$\alpha \mathtt{I}_{4\times 4} = \prod_{i=1}^{n} \mathtt{H}^{i,i+1}.$$

Taking the trace of both sides, we get the desired expression for α completing thus the proof of ii).

To prove iii), we simply remark that all the homographies $\mathtt{H}^{v_i}$ are defined up to an overall homography ambiguity. In other terms, we can replace all $\mathtt{H}^{v_i}$ by $\mathtt{H}^{v_i}\mathtt{Q}$, where $\mathtt{Q}$ is an invertible 4×4 matrix. Let $\bar{\mathtt{H}}$ be the $4n \times 4$ matrix resulting from the vertical concatenation of matrices $\mathtt{H}^{v_i}$ (satisfying conditions i) and ii) of the proposition). Let $\bar{\mathtt{H}}^\top\bar{\mathtt{H}} = \mathtt{ULU}^\top$ be the SVD of $\bar{\mathtt{H}}$. Thus, $\mathtt{L}$ is a 4×4 diagonal matrix with strictly positive diagonal entries and $\mathtt{U}$ is a 4×4 orthogonal matrix. Therefore, $\mathtt{U}$ is a homography. Setting $\mathtt{Q} = \mathtt{U}$, we get a new version of homographies $\mathtt{H}^{v_i}$ that satisfy all the conditions of Proposition 2.

References

1. Hartley, R., Zisserman., A.: Multiple view geometry in computer vision. Cambridge University, 2nd edition, (2003)